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New algorithm for calculation of Fussell-Vesely importance with application of direct partial logic derivatives

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ABSTRACT: Fussell-Vesely Importance (FVI) is one of the most commonly used importance measures. It is based on the concept of Minimal Cut Sets (MCSs), and it quantifies the contribution of a system component to system failure. In paper (Kvassay et al. 2015), the algorithm for identification of all system MCSs based on Direct Partial Logic Derivatives (DPLDs) has been considered. These MCSs are then used to compute the FVI. However, the FVI of a system component depends only on MCSs that contain the considered component, i.e. it is not necessary to identify all MCSs of the system. This implies that it is useful to develop algorithms that will identify not all system MCSs but only those in which a specific component occurs. Such algorithms can be used to compute the FVI without a priori knowledge of MCSs. In this paper, a new algorithm for this task is proposed. The algorithm is based on a relation between DPLDs and MCSs that is also considered in this paper.

1 INTRODUCTION

Principal steps in reliability engineering include estimation of system reliability and identification and quantification of situations which can cause system failure. There exist several techniques that can be used to solve these tasks. One of them is investigation of component influence on system activity. This technique is known as importance analysis.

Importance analysis uses special measures that are referred to as *Importance Measures* (IMs). These measures are used to quantify correlation between system failure and failure of its components from several points of view (Kuo & Zhu 2012). One of the most commonly known and the most commonly used IMs is *Fussell-Vesely Importance* (FVI). This measure estimates contribution of a given component to system failure. Calculation of this IM is based on a priori knowledge of *Minimal Cut Sets* (MCSs) of the considered system.

MCSs are one of the key concepts of reliability analysis. They represent minimal sets of system components whose simultaneous failure results in a failure of the system (Kuo & Zhu 2012). MCSs are used in fault tree analysis (Lim et al. 2007), in reliability analysis of two terminal networks (Gertsbakh & Shpungin 2009), etc. Many papers focus on issues related to MCSs. For example, special algorithms for reliability analysis of systems with a huge number of MCSs are considered by Cepin (2005), Choi & Cho (2007), Contini & Matuzas (2011). Other papers focus on the design of algorithms for identification or generation of MCSs of investigated system. These algorithms are developed mainly on the basis of applied fault trees (Sinnamon & Andrews 1997, Vatn 1992) or networks (Emadi & Afrakhte 2014, Rebaiaia & Ait-Kadi 2013). Another algorithm for identification of system MCSs has been considered in paper (Kvassay et al. 2015). This algorithm is based on tools of logical differential calculus, and it can be used for systems of any type (not only for systems presented as networks or for systems described by fault trees).

Logical differential calculus has been developed for analysis of dynamic properties of logic functions. *Direct Partial Logic Derivatives* (DPLDs) are essential part of this tool (Tapia et al. 1991). Their use in reliability analysis has been considered in several papers (Zaitseva 2003, Zaitseva 2012, Zaitseva et al. 2015). These papers have showed that DPLDs are very appropriate for identification of situations in which failure/repair of a system component results in system failure/repair. This fact has been used to develop algorithms for defining system boundary states and for calculation of some measures that can be used to evaluate system reliability.

In this paper, we develop ideas presented in paper (Kvassay et al. 2015) to propose an algorithm that identifies not all system MCSs but only MCSs in which a specified component is presented. Such MCSs are useful in computation of FVI of the considered component. As a result new method for calculation of this IM is developed. This method is based on DPLDs and requires no a priori knowledge of system MCSs.

2 SYSTEM STRUCTURE FUNCTION

From reliability point of view, a system can be in one of two possible states: functioning or failed. These two states can be represented by numbers 1 and 0. Furthermore, every system consists of one or more components that can also be either failed (state 0) or functioning (state 1). The dependency between states of individual system components and system state is defined by the structure function $\phi(\boldsymbol{x})$ (Zaitseva 2012):

$$\phi(\boldsymbol{x}) = \phi(x_1, x_2, \ldots, x_n)\colon \{0, 1\}^n \to \{0, 1\}, \tag{1}$$

where n denotes a number of components of the system, x_i is a variable representing state of the i-th system component and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is vector of states of system components (a state vector).

Every system component is characterized by the probabilities of:

– functioning:

$$p_i = \Pr\{x_i = 1\}, \tag{2}$$

– failure:

$$q_i = \Pr\{x_i = 0\}. \tag{3}$$

These probabilities define whether the component is available or not, therefore, they are also referred to as component availability and unavailability.

Using the structure function and availabilities (unavailabilities) of individual system components, the availability and unavailability of the whole system can be computed in the following manner (Rausand & Høyland 2004):

$$A = \Pr\{\phi(\boldsymbol{x}) = 1\}, \tag{4}$$

$$U = \Pr\{\phi(\boldsymbol{x}) = 0\}. \tag{5}$$

The availability of a system is an important characteristic because it can be viewed as a proportion of time during which the system is functioning. However, it does not allow investigating which components are the most important for the system proper work. For this task, other measures, e.g. FVI, have to be used.

In what follows, we will consider a coherent system. Such system has two principal assumptions for the structure function (Rausand & Høyland 2004): (a) the structure function (1) is monotone (the system component state decrease does not improve the system availability), and (b) the system components states are independent.

3 SYSTEM BOUNDARY STATES

The boundary state is an important concept in reliability analysis. We consider the boundary states as states for which the change of one system component causes the change of a system state. In this paper the system failure is considered. Depending on the influence of component state change to a system failure, two types of boundary states can be defined. The first type is a boundary state for which system fault depends on the change of a fixed component state. These types of boundary states are named as exact boundary states. The second type is boundary state for which the change of any system component state causes change in the system state. This type is known as *Minimal Cut Sets* (MCSs).

3.1 *Critical cut vector*

Critical cut vectors as exact boundary states have been considered in paper (Zaitseva 2003, Kvassay et al. 2015). In general, exact boundary states represent state vectors at which change of the i-th component state from s to $\tilde{s}$ causes the system performance level change from j to $\tilde{j}$ ($s, \tilde{s}, j, \tilde{j} \in \{0, 1\}$, $s \neq \tilde{s}, j \neq \tilde{j}$). The exact boundary state is defined by the exact boundary vector unambiguously.

The exact boundary states for the i-th component are calculated based on *Direct Partial Logical Derivatives* (DPLDs) with respect to variable x_i. The mathematical tool of DPLDs has been proposed in (Zaitseva 2003, Zaitseva 2012, Zaitseva & Levashenko 2013) for calculation of exact boundary states of a system. A DPLD with respect to variable x_i for the structure function (1) permits to analyze change of system state from value j to $\tilde{j}$ when the i-th component state changes from s to $\tilde{s}$ (Tapia et al. 1991):

$$\partial\phi(j \to \tilde{j})/\partial x_i(s \to \tilde{s}) = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) = j \text{ and } \phi(\tilde{s}_i, \boldsymbol{x}) = \tilde{j} \\ 0, & \text{other} \end{cases}, \tag{6}$$

where $\phi(s_i, \boldsymbol{x}) = \phi(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)$; $\phi(\tilde{s}, \boldsymbol{x}) = \phi(x_1, \ldots, x_{i-1}, \tilde{s}, x_{i+1}, \ldots, x_n)$; ($s, \tilde{s}, j, \tilde{j} \in \{0, 1\}$, $s \neq \tilde{s}, j \neq \tilde{j}$).

DPLD (6) allows investigating boundary states of a system failure (a system state in this case changes from 1 to 0) if the i-th component skips down. Therefore, this derivative allows calculating exact boundary states for the i-th component. These boundary states agree with state vectors of the form of $\boldsymbol{x} = (x_1, x_2, \ldots, x_{i-1}, 1 \to 0, x_{i+1}, \ldots, x_n)$.

For example, consider a simple service system that has been introduced in paper (Kvassay et al. 2015). This system consists of three components (n = 3) – infrastructure (component 1), service point A (component 2) and service point B (component 3). This system can be interpreted by reliability block diagram in Figure 1. The structure function of this system can be defined by Table 1 and components states probabilities are presented in Table 2.

Now, determine the exact boundary states of this service system for situations in which a failure of the first component causes system failure. For this purpose, DPLD $\partial\phi(1\to 0)/\partial x_1(1\to 0)$ can be used. The values of this derivative are in the first column in Table 3. According to this table, there are three exact boundary states for the first component. These exact boundary states can be presented as *critical cut vectors*: $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,0,1) = (0,0,1)$, $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,1,0) = (0,1,0)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1 \to 0,1,1) = (0,1,1)$. Note that, according to values of logic derivatives $\partial\phi(1\to 0)/\partial x_2(1\to 0)$ and $\partial\phi(1\to 0)/\partial x_3(1\to 0)$ presented in Table 3, this system has one critical cut vector for the second component ($\boldsymbol{x} = (x_1, x_2, x_3) = (1,1 \to 0,0) = (1,0,0)$) and one critical cut vector for the third component ($\boldsymbol{x} = (x_1, x_2, x_3) = (1,0,1 \to 0) = (1,0,0)$).

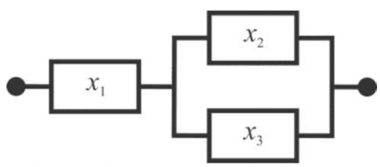

Figure 1. A simple service system.

Table 1. The structure function of the simple service system.

Components states		x_3	
x_1	x_2	0	1
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Table 2. The components states probabilities of the simple service system.

	Components states		
	x_1	x_2	x_3
p_i	0.90	0.70	0.65
q_i	0.10	0.30	0.35

DPLDs (6) allow identifying critical cut vectors of the system. These vectors agree with non-zero values of the derivative. There exist several algorithms for calculation of DPLDs. The parallel one has been proposed in paper (Zaitseva et al. 2015). For example, the investigation of the influence of the first component failure on the fault of the simple service system (Fig. 1) is performed based on DPLD $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)$, which allows calculating the system state for which the component failure causes that the system is unavailable. The parallel procedure for calculation of this derivative is shown in Figure 2 in form of flow diagram.

3.2 *Minimal cut vectors*

There is also another type of system boundary states. These boundary states are based on the concept of MCSs. The methods of reliability analysis based on MCSs have been considered in papers (Shooman 1968, Rosenberg 1996). These methods have been developed, for example, by Choi & Cho (2007), Yeh (2008) and Soh & Rai (2005).

Shooman (1968) has defined a cut set of a system as a set of system components whose simultaneous failure leads into the failure of the system (if the system has been functional). A cut set is minimal, if no component can be removed from it without losing its status as a cut set.

In the terms of the structure function, a (minimal) cut set can be interpreted by a special state vector, which is known as a (minimal) cut vector. According to the definition of a cut set, the system state for a state vector covered by a cut set is zero. Therefore, *state vector* $\boldsymbol{x}$ *is a cut vector if* $\phi(\boldsymbol{x}) = 0$.

A state vector that coincides with a MCS is a *Minimal Cut Vector* (MCV). A *cut vector* $\boldsymbol{x}$ *is minimal if* $\phi(\boldsymbol{y}) = 1$ *for any* $\boldsymbol{y} > \boldsymbol{x}$, where $\boldsymbol{y}$ is a state vector, for which $y_i \geq x_i$ ($i = 1, 2, \ldots, n$) and there exists at least one i such that $y_i > x_i$.

So, MCSs, MCVs and exact boundary states agree with system boundary states, but every of these concepts have some specifics:

– exact boundary state vectors define situations in which the skip down/repair of <u>the *i*-th component</u> causes the system failure/repair.
– MCSs agree with situations in which simultaneous failure of <u>one or more components</u> leads into the failure of the system.
– MCVs correspond to state vectors in which the renewal of <u>any failed components</u> causes the system renewal.

For example, the service system in Figure 1 with the structure function presented in Table 1 has

Table 3. DPLDs for the structure function of the simple service system.

x_2	x_3	$\frac{\partial\phi(1\to 0)}{\partial x_1(1\to 0)}$	x_1	x_3	$\frac{\partial\phi(1\to 0)}{\partial x_2(1\to 0)}$	x_1	x_2	$\frac{\partial\phi(1\to 0)}{\partial x_3(1\to 0)}$
0	0	0	0	0	0	0	0	0
0	1	1	0	1	0	0	1	0
1	0	1	1	0	1	1	0	1
1	1	1	1	1	0	1	1	0

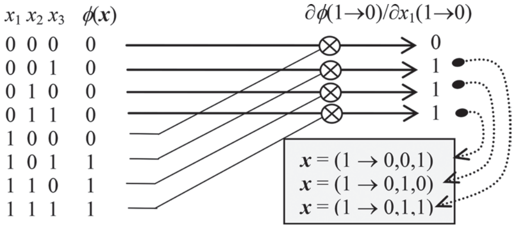

Figure 2. Calculation of the DPLD $\partial\phi(1\to 0)/\partial x_1(1\to 0)$.

Table 4. MCVs of the simple service system.

Critical cut vector	MCVs
(0,0,1)	
(0,1,0)	(0,1,1)
(0,1,1)	(1,0,0)
(1,0,0)	
(1,0,0)	

2 MCVs $\boldsymbol{x} = (x_1, x_2, x_3) = (0,1,1)$ and $\boldsymbol{x} = (x_1, x_2, x_3) = (1,0,0)$.

In Table 4, exact boundary state vectors (critical cut vectors) and MCVs are presented for this system. And we can see that these two groups include different state vectors. Therefore, there are two types of boundary states of the system that agree with different conditions and that are defined by different types of boundary states vectors.

Therefore, the algorithms for calculation of different boundary states of the system are not similar and are based on analysis of specific condition of the system criticality. These types of boundary states are used for calculation of different reliability indices. IMs are one group of these indices. In papers (Zaitseva 2013, Zaitseva et al. 2015) the application of exact boundary states for calculation of structural importance, Birnbaum importance and criticality importance has been considered. But, the calculation of FVI based on this approach is not possible because FVI is defined based on the concept of MCSs or MCVs (Kuo & Zhu 2012).

4 FUSSELL-VESELY IMPORTANCE BASED ON LOGICAL DIFFERENTIAL CALCULUS

4.1 *Fussell-Vesely importance and minimal cut vectors*

The FVI is one of the most commonly used IMs. It quantifies the contribution of a system component to system failure. This IM is based on MCSs and, taking into account the relation between MCSs and MCVs, it can be computed as follows (Kvassay et al. 2015):

$$FVI_i = \frac{\Pr\{\exists \mathrm{MCV}(0_i) \in \mathrm{MCVs};\, \boldsymbol{x} \le \mathrm{MCV}(0_i)\}}{U}, \tag{7}$$

where $MCVs$ is a set of all MCVs of the system, $\mathrm{MCV}(0_i)$ is a MCV in which $x_i = 0$, the event $\{\exists \mathrm{MCV}(0_i) \in \mathrm{MCVs};\, \boldsymbol{x} \le \mathrm{MCV}(0_i)\}$ means that there is at least one MCV with $x_i = 0$ that is greater than or equal to an arbitrary state vector $\boldsymbol{x}$, and U is system unavailability (5).

The structural version of the FVI of component i (Kuo & Zhu 2012) can be defined as the relative number of cut vectors that are less than or equal to at least one MCV containing component i in state 0:

$$SFVI_i = \frac{\left|\{\exists \mathrm{MCV}(0_i) \in \mathrm{MCVs};\, \boldsymbol{x} \le \mathrm{MCV}(0_i)\}\right|}{\left|\{\boldsymbol{x} : \phi(\boldsymbol{x}) = 0\}\right|}, \tag{8}$$

Kvassay et al (2015) have proposed an algorithm for calculation of the FVI based on DPLDs. This algorithm requires computation of conjunction of a special type of DPLDs that are called expanded DPLDs. Another way is to develop a definition of the FVI that will be based directly on DPLD $\partial\phi(0\to 1)/\partial x_i(0\to 1)$ and then compute the FVI without the need to compute DPLDs conjunction and identify MCVs from it. Now, we concentrate on this approach.

4.2 *Fussell-Vesely importance based on minimal cut vectors and direct partial logic derivatives*

Consider a system of n components and DPLD $\partial\phi(0\to 1)/\partial x_i(0\to 1)$ for some $i \in \{1, 2, \ldots, n\}$.

Specially, this derivative is defined only for state vectors $(0_i, \boldsymbol{x})$ of the structure function, therefore, nonzero values of expression $\bar{x}_i\, \partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ agree with critical cut vectors for component i.

Next, recall that a MCV corresponds to a situation in which a repair of any failed component results in system repair. Therefore, a MCV can be defined as a cut vector that is critical for all components that are failed, i.e. that are in state 0. According to definition (7) of the FVI of component i, only MCVs that contain component i in state 0 are needed to compute the FVI based on MCVs. This implies that definition (7) requires only MCVs that have form of $(0_i, \boldsymbol{x})$. According to the meaning of a MCV, these MCVs are also critical cut vectors for component i. Based on the previous paragraph, these critical cut vectors correspond to nonzero elements of expression $\bar{x}_i\, \partial\phi(0 \to 1)/\partial x_i(0 \to 1)$. Now, we only need to identify which elements (state vectors) of this expression are MCVs.

Consider two critical cut vectors $(0_i, \boldsymbol{x})$ and $(0_i, \boldsymbol{y})$ for component i and assume that $(0_i, \boldsymbol{x}) < (0_i, \boldsymbol{y})$. The fact that these vectors are cut vectors implies that $\phi(0_i, \boldsymbol{x}) = 0$ and $\phi(0_i, \boldsymbol{y}) = 0$. However, this also implies that the state vector $(0_i, \boldsymbol{x})$ cannot be a MCV because there exists at least one state vector, i.e. the state vector $(0_i, \boldsymbol{y})$, that is greater than the state vector $(0_i, \boldsymbol{x})$ and that is also a cut vector. Following this fact, if we know all critical cut vectors for component i, then MCVs can only be the maximal ones. Please note that a critical cut vector $(0_i, \boldsymbol{x})$ is maximal if no other critical cut vector $(0_i, \boldsymbol{y})$ that satisfies relation $(0_i, \boldsymbol{x}) < (0_i, \boldsymbol{y})$ exists, e.g. the simple service system in Figure 1 has 3 critical cut vectors for the first component, i.e. (0,0,1), (0,1,0), and (0,1,1) (see Table 4), and the maximal one is (0,1,1) because both others are less than this one (in terms of relation "<" defined on the set of all system state vectors). Therefore, every MCV of the form $(0_i, \boldsymbol{x})$ corresponds to a maximal critical cut vector for this component.

Next, we need to show that every maximal critical cut vector for component i also corresponds to a MCV of the form of $(0_i, \boldsymbol{x})$, i.e. that there exists one-to-one correspondence between MCVs of the form of $(0_i, \boldsymbol{x})$ and maximal critical cut vectors for component i. For this purpose, assume that critical cut vector $(0_i, 0_k, \boldsymbol{x})$ is a maximal critical cut vector for component i but not a MCV because $\phi(0_i, 1_k, \boldsymbol{x}) = 0$. This implies that $\phi(0_i, 0_k, \boldsymbol{x}) = 0$ and $\phi(1_i, 0_k, \boldsymbol{x}) = 1$. Next, the coherency implies that $\phi(1_i, 1_k, \boldsymbol{x}) = 1$. This means that state vector $(0_i, 1_k, \boldsymbol{x})$ is a critical cut vector. But, this is a contradiction with the assumption that state vector $(0_i, 0_k, \boldsymbol{x})$ is a maximal critical cut vector for component i. Therefore, every maximal critical cut vector for component i also corresponds to a MCV of the form of $(0_i, \boldsymbol{x})$. These results can be concluded in the form of the next theorem and corollary.

Theorem. A state vector $(0_i, \boldsymbol{x})$ is a MCV iff it is a maximal state vector for which expression $\bar{x}_i\, \partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ has nonzero value.

Corollary. Consider a coherent system of n components with the structure function $\phi(\boldsymbol{x})$. A set of all MCVs of the form of $(0_i, \boldsymbol{x})$ corresponds to the following set:

$$\max\left\{\underset{\boldsymbol{x}\in\{0,1\}^n}{\operatorname{argone}}\left(\bar{x}_i \frac{\partial\phi(0 \to 1)}{\partial x_i(0 \to 1)}\right)\right\}. \tag{9}$$

Please note that the function argone(.) in (9) identifies all state vectors for which the expression $\bar{x}_i\, \partial\phi(0 \to 1)/\partial x_i(0 \to 1)$ has nonzero value, i.e. critical cut vectors for component i, while the function $\max\{.\}$ takes the maximal ones from them in terms of relation "<" defined on the set of all system state vectors.

Using the Corollary, the FVI (7) can be rewritten as follows:

$$FVI_i = \frac{\Pr\left\{\exists \boldsymbol{y} \in \max\left\{\underset{\boldsymbol{x}\in\{0,1\}^n}{\operatorname{argone}}\left(\bar{x}_i \frac{\partial\phi(0 \to 1)}{\partial x_i(0 \to 1)}\right)\right\}; \boldsymbol{x} \le \boldsymbol{y}\right\}}{U}. \tag{10}$$

Next, define a special function $f_i^-(\boldsymbol{x})$ of n variables where $i \in \{1, 2, \ldots, n\}$ as follows:

$$f_i^-(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \exists \boldsymbol{y} \in \max\left\{\underset{\boldsymbol{x}\in\{0,1\}^n}{\operatorname{argone}}\left(\bar{x}_i \frac{\partial\phi(0 \to 1)}{\partial x_i(0 \to 1)}\right)\right\}; \boldsymbol{x} \le \boldsymbol{y} \\ 0 & \text{else} \end{cases}, \tag{11}$$

Clearly, this function has value 1 for a vector $\boldsymbol{x}$ if at least one MCV that is greater than or equal to the state vector $\boldsymbol{x}$ exists in the set (9) of all MCVs that contain component i in state 0, and it has value 0 if no such MCV exists in the set. Definition (11) implies that the function $f_i^-(\boldsymbol{x})$ has a property that $f_i^-(\boldsymbol{x}) \ge f_i^-(\boldsymbol{y})$ if $\boldsymbol{x} < \boldsymbol{y}$, therefore, it can be interpreted as a negative i.e. non-increasing, Boolean function. It is also possible to show that definition (10) is equivalent with the next one:

$$f_i^-(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \exists \boldsymbol{y} \in \left\{\underset{\boldsymbol{x}\in\{0,1\}^n}{\operatorname{argone}}\left(\bar{x}_i \frac{\partial\phi(0 \to 1)}{\partial x_i(0 \to 1)}\right)\right\}; \boldsymbol{x} \le \boldsymbol{y} \\ 0 & \text{else} \end{cases}, \tag{12}$$

which states that the function $f_i^-(\boldsymbol{x})$ has value 1 for a state vector $\boldsymbol{x}$ if and only if the expression $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$ is nonzero for at least one state vector that is greater than or equal to the state vector $\boldsymbol{x}$. Formula (12) corresponds to the definition of the minimal negative function for Boolean function $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$. (Please note that the minimal negative function for a Boolean function is a non-increasing Boolean function that can be created from the considered function in such a way that we change value of the considered function to value 1 in the minimal number of points at which the considered function takes value 0.) Therefore, the function $f_i^-(\boldsymbol{x})$ can be expressed as follows:

$$f_i^-(\boldsymbol{x}) = \mathrm{MNF}\left(\bar{x}_i \frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}\right), \tag{13}$$

where MNF(.) denotes the minimal negative Boolean function created from a Boolean function occurring in the argument. If we look at the numerator in definition (10) of the FVI, then we can recognize that it corresponds to the probability that the minimal negative Boolean function for the expression $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$ is nonzero. Therefore, the FVI can be expressed based on DPLDs as follows:

$$FVI_i = \frac{\Pr\left\{\mathrm{MNF}\left(\bar{x}_i \frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}\right)=1\right\}}{U} = \frac{q_i \Pr\left\{\mathrm{MNF}\left(\frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}\right)=1\right\}}{U} \tag{14}$$

Using the similar approach, it can be shown that the SFVI (8) can be defined as a relative number of cut vectors for which the Boolean expression $\mathrm{MNF}\left(\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)\right)$ is nonzero:

$$SFVI_i = \frac{\left|\left\{\boldsymbol{x}{:}\mathrm{MNF}\left(\bar{x}_i \frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}\right)=1\right\}\right|}{\left|\left\{\boldsymbol{x}{:}\phi(\boldsymbol{x})=0\right\}\right|} = \frac{\mathrm{TD}\left(\mathrm{MNF}\left(\bar{x}_i \frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}\right)\right)}{\mathrm{TD}\left(\overline{\phi(\boldsymbol{x})}\right)} \tag{15}$$

where $\{\boldsymbol{x}{:}\phi(\boldsymbol{x})=0\}$ is a set of all system cut vectors, $\{\boldsymbol{x}:\mathrm{MNF}(\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1))=1\}$ is a set of all state vectors for which the minimal negative Boolean function for $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$ is nonzero (according to the previous text, every state vector of the minimal negative Boolean function corresponds to a cut vector of the system), $|.|$ denotes the size of the considered set, and TD(.) denotes the truth density of the argument interpreted as a Boolean function (it corresponds to a relative number of points in which the Boolean function takes value 1).

4.3 *Hand calculation example*

As an example to illustrate the calculation of the FVI measures using DPLDs, consider the simple service system in Figure 1, whose structure function is in Table 1. This function can be interpreted as a Boolean function $\phi(\boldsymbol{x}) = x_1x_2 \vee x_1x_3$. In the firs step, DPLDs $\partial\phi(0\to 1)/\partial x_i(0\to 1)$ have to be computed for all system components, i.e. for $i = 1, 2, 3$, to find values of the FVI of individual system components. These derivatives are in the second row of Table 5. Next, identify expressions $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$. This is done simple by expanding individual DPLDs using literal $\bar{x}_i$ (the third row of Table 5). When these expressions are known, then transform them in the form of the minimal negative Boolean functions (the fourth row of Table 5), which are used in definitions (14) and (15) of the FVI and SFVI respectively. For example, consider component 1 of the system. DPLD $\partial\phi(0\to 1)/\partial x_1(0\to 1)$ is calculated as $x_2 \vee x_3$, and its conjunction with literal $\bar{x}_1$ has the form of $\bar{x}_1x_2 \vee \bar{x}_1x_3$. Since this expression corresponds to a disjunctive normal form, the corresponding minimal negative Boolean function can be obtained simply by removing all positive literals, i.e. x_2 and x_3. So, the minimal negative Boolean function for the expression $\bar{x}_1\partial\phi(0\to 1)/\partial x_1(0\to 1)$ has the form of $\bar{x}_1$.

When the minimal negative Boolean function for the expression $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$ is known, then the SFVI of the i-th system component can be computed as a quotient of the truth densities of the minimal negative Boolean function and the negation of the structure function. In the considered system, the negation of the structure function has the form of $\bar{x}_1 \vee \bar{x}_2\bar{x}_3$ and, therefore, its truth density is 0.625. Next, for example, the truth density of the minimal negative Boolean function for expression $\bar{x}_1\partial\phi(0\to 1)/\partial x_1(0\to 1)$ is 0.5. This implies that the $\mathrm{SFV_cI}$ of the 1-st component is computed as 0.5/0.625 = 0.8. For other system components, the truth densities of minimal negative functions for expressions $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$ are presented in Table 5 and the final values of the SFVI are in Table 6. If we have information about availabilities of individual system components (Table 2), then we can also compute the FVI of individual system components. This computation is also based on the minimal negative Boolean functions for expressions $\bar{x}_i\,\partial\phi(0\to 1)/\partial x_i(0\to 1)$. For example,

Table 5. DPLDs, the corresponding minimal negative functions, and their truth densities for the system in Figure 1.

	Components		
	x_1	x_2	x_3
$\frac{\partial\phi(0\to1)}{\partial x_i(0\to1)}$	$x_2 \vee x_3$	$x_1\bar{x}_3$	$x_1\bar{x}_2$
$\bar{x}_i\frac{\partial\phi(0\to1)}{\partial x_i(0\to1)}$	$\bar{x}_1x_2 \vee \bar{x}_1x_3$	$x_1\bar{x}_2\bar{x}_3$	$x_1\bar{x}_2\bar{x}_3$
$\mathrm{MNF}\left(\bar{x}_i\frac{\partial\phi(0\to1)}{\partial x_i(0\to1)}\right)$	$\bar{x}_1$	$\bar{x}_2\bar{x}_3$	$\bar{x}_2\bar{x}_3$
$\mathrm{TD}\left(\mathrm{MNF}\left(\bar{x}_i\frac{\partial\phi(0\to1)}{\partial x_i(0\to1)}\right)\right)$	0.50	0.25	0.25

Table 6. Fusell-Vesely's measures for the system.

	Components		
	x_1	x_2	x_3
$\Pr\left\{\mathrm{MNF}\left(\bar{x}_i\frac{\partial\phi(0\to1)}{\partial x_i(0\to1)}\right)=1\right\}$	0.1000	0.1050	0.1050
FVI	0.5141	0.5398	0.5398
SFVI	0.80	0.40	0.40

if we want to quantify the contribution of the 1-st component to system failure, then we can calculate it as a quotient of the probability that the minimal negative Boolean function for the expression $\bar{x}_1\partial\phi(0\to1)/\partial x_1(0\to1)$ is nonzero, i.e. 0.1, and the probability that the system is unavailable, i.e. 0.1945. Therefore, the FVI of component 1 has value 0.5141. Similarly, we can compute the FVI for other two components of the system (see Table 6). The values of the FVI presented in Table 6 indicate that the 1-st component has the most contribution to system failure from topological point of view while if we take into account the availabilities of individual system components, then the 2-nd and 3-rd component contribute more than the 1-st one. These results are caused by two facts. Firstly, from topological point of view, a failure of component 1 ensures that the system will be unavailable regardless of states of other two components while a failure of the 2-nd (3-rd) component can contribute to system failure if and only if the 3-rd (2-nd) component is unavailable. On the other hand, the 1-st component is more reliable than other two components together because its failure is less probable (its unavailability is 0.1) than a failure of other two components (their joint unavailability is 0.105). Therefore, if the availabilities of individual system components are considered, then the 2-nd and 3-rd component have the most contribution to system failure.

5 CONCLUSIONS

The principal result of this paper is development of mathematical approach of DPLDs for the calculation of MCVs. Mathematical background for this development is considered and used for new definition of FVI measure (7) and (8). The new definitions (14) and (15) are based on DPLDs and they require no a priori knowledge of MCSs or MCVs. The proposed method is explained in the hand calculation example (it is the system in Fig. 1). The next step of the investigation will be comparison of this method with other ones that are used for calculation of FVI, e.g. Kuo & Zhu (2012) and Kvassay et al. (2015). The preliminary examination of algorithm presented in (Kvassay et al. 2015) and new algorithm in this paper permits supposing that time complexity of new algorithm for analysis of the system will be less in n times approximately (n is number of the system components) because the algorithm in (Kvassay et al. 2015) is based on the calculation of DPBDs for all variables of the system structure function. New algorithm is based on the computation of only one derivative for analysed variable of the structure function.

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